CONFIGURATION SPACES ARE NOT HOMOTOPY INVARIANT

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Abstract. We present a counterexample to the conjecture on the homotopy invariance of configuration spaces. More precisely, we consider the lens spaces $L_{7,1}$ and $L_{7,2}$, and prove that their configuration spaces are not homotopy equivalent by showing that their universal coverings have different Massey products.

1. Introduction

The configuration space $F_n(M)$ of pairwise distinct n-tuples of points in a manifold M has been much studied in the literature. Levitt reported in [4] as "long-standing" the following

Conjecture 1.1. The homotopy type of $F_n(M)$, for M a closed compact smooth manifold, depends only on the homotopy type of M.

There was some evidence in favor: Levitt proved that the loop space $\Omega F_n(M)$ is a homotopy invariant of M. Recently Aouina and Klein [1] have proved that a suitable iterated suspension of $F_n(M)$ is a homotopy invariant. For example the double suspension of $F_2(M)$ is a homotopy invariant. Moreover $F_2(M)$ is a homotopy invariant when M is 2-connected (see [4]). A rational homotopy theoretic version of this fact appears in [3]. On the other hand there is a similar situation suggesting that the conjecture might fail: the Euclidean configuration space $F_3(\mathbb{R}^n)$ has the homotopy type of a bundle over S^{n-1} with fiber $S^{n-1} \vee S^{n-1}$ but it does not split as a product in general [6]. However the loop spaces of $F_3(\mathbb{R}^n)$ and of the product $S^{n-1} \times (S^{n-1} \vee S^{n-1})$ are homotopy equivalent and also the suspensions of the two spaces are homotopic.

Lens spaces provide handy examples of manifolds which are homotopy equivalent but not homeomorphic. The first of these examples are $L_{7,1}$ and $L_{7,2}$. We show that their configuration spaces $F_2(L_{7,1})$ and $F_2(L_{7,2})$ are not homotopy equivalent. After recalling some definition, we will describe the universal coverings of these configuration spaces. Such coverings can be written as bundles with same base and fiber, but the first splits and the second does not. We will show that Massey products are all zero in the first case, while there exists a nontrivial Massey product in the second case.

This means that $F_2(L_{7,1})$ is not homotopy equivalent to $F_2(L_{7,2})$. Finally we will extend this result by showing that $F_n(L_{7,1})$ is not homotopy equivalent to $F_n(L_{7,2})$ for any $n \geq 2$. The same result holds for unordered configuration spaces.

2. Configuration spaces of lens spaces

The lens spaces are 3-dimensional oriented manifolds defined as

$$L_{m,n} := S^3/\mathbb{Z}_m = \left\{ (x_1, x_2) \in \mathbb{C} \times \mathbb{C} \,\middle|\, |x_1|^2 + |x_2|^2 = 1 \right\} /\mathbb{Z}_m$$

where the group action is defined by $\zeta((x_1, x_2)) = (e^{2\pi i/m}x_1, e^{2\pi in/m}x_2)$, and ζ is the generator of $\mathbb{Z}_m$. It is known (see e.g. [7]) that $L_{7,1}$ and $L_{7,2}$ are homotopy equivalent, though not homeomorphic.

For any topological space M, let $F_n(M)$ be the configuration space of n pairwise distinct points in M, namely $F_n(M) := M^n \setminus (\bigcup \Delta)$ where $\bigcup \Delta$ is the union of all diagonals. We first want to compute the fundamental group of $F_2(L_{7,1})$ and $F_2(L_{7,2})$. Observe that S^3 is the universal covering of $L_{7,j}$, for $j = 1, 2$, and therefore the fundamental group of $L_{7,j}$ is $\mathbb{Z}_7$. Then $\pi_1(F_2(L_{7,j})) = \mathbb{Z}_7 \times \mathbb{Z}_7$ because $\pi_1(L_{7,j} \times L_{7,j}) = \mathbb{Z}_7 \times \mathbb{Z}_7$ and removing the diagonal, which is a codimension 3 manifold, does not change the fundamental group.

The universal coverings $\widetilde{F}_2(L_{7,1})$ and $\widetilde{F}_2(L_{7,2})$ are the so-called "orbit configuration spaces" and are given by pairs of points (x, y) of S^3 which don't lie on the same orbit, i.e., $x \neq g(y)$ for any $g \in \mathbb{Z}_7$.

In the rest of the paper we identify $\mathbb{Z}_7$ to the group of 7th complex roots of unity, and we use the symbol ζ^t, $t \in \mathbb{R}$, to denote the complex number $e^{2\pi i t/7}$.

The first universal covering has a simple structure, namely we have the following

Proposition 2.1. *$\widetilde{F}_2(L_{7,1})$ is homotopy equivalent to $\vee_6 S^2 \times S^3$.*

Proof. It is convenient to interpret S^3 as the space of quaternions of unitary norm. Then the action of $\mathbb{Z}_7$ on $S^3 = \widetilde{L_{7,1}}$ is the left translation by the subgroup $\mathbb{Z}_7 \subset \mathbb{C} \subset \mathbb{H}$. We define a map $\widetilde{F}_2(L_{7,1}) \to (S^3 \setminus \mathbb{Z}_7) \times S^3$ by sending (x, y) to (xy^{-1}, y). This is a homeomorphism since $x \neq \zeta^k(y) = \zeta^k y$ is equivalent to $xy^{-1} \neq \zeta^k$ for any 7th root of unity $\zeta^k, k \in \{0, \dots, 6\}$. Finally we observe that S^3 minus a point is $\mathbb{R}^3$ and hence $S^3 \setminus \mathbb{Z}_7$ is homotopic to the wedge of six 2-dimensional spheres. $\square$

3. Massey products

We briefly recall the definition of Massey products for a topological space X (see [5]). Let $x, y, z \in H^*(X)$ such that $x \cup y = y \cup z = 0$. If we choose singular cochain representatives $\bar{x}, \bar{y}, \bar{z} \in C^*(X)$ then we have that $\bar{x} \cup \bar{y} = dZ$ and $\bar{y} \cup \bar{z} = dX$ for some cochains Z and Y. Notice that

$$d(Z \cup \bar{z} - (-1)^{\deg(x)} \bar{x} \cup X) = (\bar{x} \cup \bar{y} \cup \bar{z} - \bar{x} \cup \bar{y} \cup \bar{z}) = 0,$$

and hence we can define $\langle x, y, z \rangle$ to be the cohomology class of $Z \cup \bar{z} - (-1)^{\deg(x)} \bar{x} \cup X$. Since the choice of Z and X is not unique, the Massey product $\langle x, y, z \rangle$ is well defined only in $H^*(X)/\langle y, z \rangle$ where $\langle y, z \rangle$ is the ideal generated by y and z. Clearly Massey products are homotopy invariants. A rational homotopy theoretic version of the following definition is in [2].

Definition 3.1. A space X is formal if the singular cochain complex $C^*(X)$ is quasi-isomorphic to $H^*(X)$ as augmented differential graded ring.

This means there is a zig-zag of homomorphisms inducing isomorphism in cohomology and connecting $H^*(X)$ and $C^*(X)$. It is easy to see that spheres are formal. Moreover wedges and products of formal spaces are formal. By construction all Massey products on the cohomology of a formal space vanish. This in turn implies the following result

Proposition 3.2. *All Massey products in the cohomology of $\widetilde{F}_2(L_{7,1})$ are trivial.*

We deduce that in order to prove that $\widetilde{F}_2(L_{7,1})$ and $\widetilde{F}_2(L_{7,2})$ are not homotopy equivalent, we only need to construct a nontrivial Massey product in the cohomology $\widetilde{F}_2(L_{7,2})$.

4. Nontrivial Massey product for $\widetilde{F}_2(L_{7,2})$

The projection onto the first coordinate gives $\widetilde{F}_2(L_{7,2})$ the structure of a bundle over S^3 with fiber $S^3 \setminus \mathbb{Z}_7 \simeq \vee_6 S^2$ that admits a section. It follows that the Serre spectral sequence collapses and the cohomology ring splits as a tensor product, so that it does not detect the nontriviality of the bundle. In particular we have that $H^2(\widetilde{F}_2(L_{7,2})) \cong \mathbb{Z}^6$ and $H^4(\widetilde{F}_2(L_{7,2})) = 0$. This in turn implies that the Massey product of any triple in H^2 is well defined.

We want to compute Massey products "geometrically", namely using intersection theory on the Poincaré dual cycles as in [5].

Let us define the embedded "diagonal" 3-spheres $\Delta_k \subset S^3 \times S^3$, for $k = 0, \dots 6$, by $\Delta_k := \{(x, \zeta^k(x)) \,|\, x \in S^3\}$. Clearly Δ_0 is the standard diagonal. The space $\widetilde{F}_2(L_{7,2})$ is the complement of the union of the diagonals

$$\widetilde{F}_2(L_{7,2}) = (S^3 \times S^3) \setminus \left(\coprod_{k=0}^{6} \Delta_k \right).$$

By Poincaré duality we have the isomorphism

$$H^p\left((S^3 \times S^3) \setminus \left(\coprod_{k=0}^{6} \Delta_k \right) \right) \cong H_{6-p}\left(S^3 \times S^3, \left(\coprod_{k=0}^{6} \Delta_k \right) \right).$$

Under this identification the cup product in cohomology corresponds to the intersection product in homology.

We observe that there exists an isotopy $\mathcal{H}_k : S^3 \times [0,1] \to S^3 \times S^3$ (where k is considered mod 7) defined by $\mathcal{H}_k((x_1, x_2), t) = ((x_1, x_2), (\zeta^{k-1+t} x_1, \zeta^{2(k-1+t)} x_2))$. The images of $\mathcal{H}_k$ at times 0 and 1 are respectively Δ_{k-1} and Δ_k, and the full image of $\mathcal{H}_k$ is a submanifold $A_k \subset S^3 \times S^3$ which represents an element in $H_4\left(S^3 \times S^3, \left(\coprod_{k=0}^{6} \Delta_k \right) \right)$ Poincaré dual to a class $a_k \in H^2(\widetilde{F}_2(L_{7,2}))$. By using the Mayer-Vietoris sequence one can easily see that the classes a_k span $H^2(\widetilde{F}_2(L_{7,2}))$ under the relation $\sum_{k=0}^{6} a_k = 0$. The main result of the paper is the following

Theorem 4.1. *The Massey product $\langle a_4, a_1, a_2 + a_6 \rangle$ contains the class $a_2 \cup \iota$ and hence is nontrivial.*

Proof. It is easy to check that A_k intersects only A_{k+3} and A_{k+4} where again k is considered mod 7. Hence in the computation of $\langle a_4, a_1, a_2 + a_6 \rangle$ we must check the following

Lemma 4.2. *The submanifolds A_1 and A_4 intersect transversally and*

$$S^1 \times [0,1] \cong A_1 \cap A_4 = \left\{ \left((0, x_2), (0, \zeta^\lambda x_2)\right) \middle| \, |x_2| = 1, \, \lambda \in [0,1] \right\}.$$

Proof. We only need to verify that the tangent spaces to A_1 and A_4 at the point $\left((0, x_2), (0, \zeta^\lambda x_2)\right)$ span a six dimensional vector space. Recall that we are representing points in S^3 as elements (x_1, x_2) in $\mathbb{C} \times \mathbb{C}$ such that $|x_1|^2 + |x_2|^2 = 1$, and hence tangent vectors at $(0, x_2)$ are real linear combinations of the vectors $(1, 0)$,

$(i,0)$ and $(0,ix_2)$. These immediately give rise to the following tangent vectors to A_1 at $\big((0,x_2),(0,\zeta^\lambda x_2)\big)$:

$$\Big((1,0),(\zeta^{\lambda/2},0)\Big), \quad \Big((i,0),(i\zeta^{\lambda/2},0)\Big), \quad \big((0,ix_2),(0,i\zeta^\lambda x_2)\big)$$

and to the following tangent vectors to A_4 at the same point:

$$\Big((1,0),(-\zeta^{\lambda/2},0)\Big), \quad \Big((i,0),(-i\zeta^{\lambda/2},0)\Big), \quad \big((0,ix_2),(0,i\zeta^\lambda x_2)\big).$$

Finally consider the path in $A_1 \cap A_4$ given by

$$s \mapsto \big((0,x_2),(0,\zeta^{\lambda+s}x_2\big).$$

Its derivative for $s=0$ gives, up to a scalar factor, the vector $\big((0,0),(0,i\zeta^\lambda x_2)\big)$. By a simple inspection one sees that the linear space spanned by these vectors is six dimensional. $\square$

Let us consider the closed 2-disc

$$D_2 = \{(r,x)\,|\,0 \le r \le 1, r^2+|x|^2=1,\, x\in\mathbb{C}\} \subset S^3.$$

Lemma 4.3. *The intersection $A_1 \cap A_4$ is the relative boundary of the 3-manifold*

$$D_2\times[0,1] \cong X_{14} := \Big\{\,\big((r,x),(\zeta^{4t}r,\zeta^t x)\big)\Big|\,(r,x)\in D_2,\ 0\le t\le 1\Big\}.$$

Proof. The pieces of the boundary of X_{14} correspond to $r=0$, $t=0$ and $t=1$. Clearly $\partial_{r=0}X_{14} = A_1\cap A_4$. If we now show that the other pieces belong to one of the diagonals Δ_k, the Lemma is proved. Since $\zeta^k=\zeta^{k+7}$ we have

$$\partial_{t=0}X_{14} = \{((r,x),(r,x))\}\subset\Delta_0$$
$$\partial_{t=1}X_{14} = \big\{\big((r,x),(\zeta^4 r,\zeta x)\big)\big\}\subset\Delta_4.$$

$\square$

The next step is to find the intersection of X_{14} with A_2 and A_6. We observe that the inclusion $S^3\to S^3\times S^3$ sending x to $(1,x)$ represents the generator of $H_3\left(S^3\times S^3,\coprod_{k=0}^6\Delta_k\right)\cong\mathbb{Z}$ Poincaré dual to a class $\iota\in H^3(\widetilde{F}_2(L_{7,2}))\cong\mathbb{Z}$.

Lemma 4.4. *The manifolds X_{14} and A_6 do not intersect. Moreover X_{14} and A_2 intersect transversally and $X_{14}\cap A_2 = A_2\cap S^3$ is Poincaré dual to the class $a_2\cup\iota$.*

Proof. The intersection of X_{14} with A_6 is given by the solution to the system of equations

$$\begin{cases}\zeta^{4t}r=\zeta^{5+s}r\\ \zeta^t x_2=\zeta^{10+2s}x_2\end{cases}$$

for $0\le r\le 1$, $r^2+|x|^2=1$, $0\le t\le 1$ and $0\le s\le 1$. If we equate the exponents of the ζ's in the first and in the second equation we immediately see that there are no solutions for $0\le t\le 1$.

The intersection of X_{14} with A_2 is given by the solution to the system of equations

$$\begin{cases}\zeta^{4t}r=\zeta^{1+s}r\\ \zeta^t x=\zeta^{2+2s}x\end{cases}$$

which has solutions $\big((1,0),(\zeta^{1+s},0)\big)$ where $0\le s\le 1$. In fact, from the second equation we get the equation $t=2+2s \pmod 7$, which has no solution for $0\le t\le 1$. Therefore we must have $x=0$ and $r=1$. From the first equation we have that

$\zeta^{4t} = \zeta^{1+s}$ which implies $t = (1+s)/4$. Therefore $X_{14} \cap A_2$ is a path connecting Δ_1 with Δ_2 which equals $A_2 \cap S^3$.

Finally we have to check transversality for X_{14} and A_2. By repeating the arguments of Lemma 4.2, we deduce that the tangent space to A_2 at $((1,0),(\zeta^{1+s},0)) = ((1,0),(\zeta^{4t},0))$ is spanned by $((i,0)(i\zeta^{1+s},0))$, $((0,1),(0,\zeta^{2+2s}))$, $((0,i),(0,i\zeta^{2+2s}))$ and $((0,0),(i\zeta^{1+s},0))$ while the tangent space to X_{14} at the same point is spanned by $((0,1),(0,\zeta^t))$, $((0,i),(0,i\zeta^t))$ and $((0,0),(i\zeta^{4t},0))$. These vectors clearly span a six dimensional space. $\square$

This concludes the proof since $a_2 \cup \iota$ does not belong to the subspace generated by $a_4 \cup \iota$ and $(a_2 + a_6) \cup \iota$ in

$$H^5(\widetilde{F}_2(L_{7,2})) = \langle a_k \cup \iota |\, k = 0, \dots, 6\rangle \Big/ \sum_{k=0}^{6} a_k \cup \iota \; .$$

$\square$

5. Generalizations

We extend our result to the n points configuration space, namely we have that $F_n(L_{7,1})$ is not homotopic to $F_n(L_{7,2})$. The universal covering $\widetilde{F}_n(L_{7,j})$ is the orbit configuration space of n-tuples of points in S^3 lying in pairwise distinct $\mathbb{Z}_7$-orbits. The forgetful map $(x_1, \dots, x_n) \mapsto (x_1, x_2)$ defines a bundle $\widetilde{F}_n(L_{7,j}) \to \widetilde{F}_2(L_{7,j})$ which admits a section. For example the values $x_3, \dots, x_n$ of the section are pairwise distinct points very close to 1 multiplied by x_1.

By naturality we deduce that $\widetilde{F}_n(L_{7,2})$ has a nontrivial Massey product on H^2. On the other hand right multiplication by x_1^{-1} induces a product decomposition $\widetilde{F}_n(L_{7,1}) = S^3 \times Y_{n-1}$, where Y_{n-1} is the $n-1$ points orbit configuration space of the $\mathbb{Z}_7$-space $S^3 \setminus \mathbb{Z}_7$. The forgetful map picking the first coordinate defines a bundle $Y_2 \to S^3 \setminus \mathbb{Z}_7$ having as fiber S^3 with 14 points removed. By iterating this procedure we find a tower of fibrations expressing Y_{n-1} as twisted product, up to homotopy, of the wedges of spheres $\vee_6 S^2, \vee_{13} S^2$, and so on. The additive homology of Y_{n-1} splits as tensor product of the homology of the factors, by the Serre spectral sequence. In particular there is a map $\vee_{(n-1)(7n-2)/2} S^2 \to Y_{n-1}$ inducing isomorphism on H_2. The product map $S^3 \times \vee_{(n-1)(7n-2)/2} S^2 \to \widetilde{F}_n(L_{7,1})$ induces isomorphism on the cohomology groups H^2, H^3, H^5. Thus all Massey products on elements of $H^2(\widetilde{F}_n(L_{7,1}))$ must vanish.

The unordered configuration space $C_n(L_{7,j}) = F_n(L_{7,j})/\Sigma_n$ has as fundamental group the wreath product $\Sigma_n \wr \mathbb{Z}_7$ and has the same universal cover as the ordered configuration space. It follows that also all unordered configuration spaces are not homotopy invariant.

Our approach shows that infinite other pairs of homotopic lens spaces have non homotopic configuration spaces. It might be interesting to study whether the homotopy type of configuration spaces distinguishes up to homeomorphism all lens spaces.

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